Combinatorics 2008

June 22-28

Riordan arrays and the reciprocation of series: two approaches.*

Ana Luzón

Departamento de Matemática Aplicada a los Recursos Naturales. E.T.S.I. Montes. Universidad Politécnica de Madrid. 28040-Madrid, SPAIN

anamaria.luzon@upm.es

*Manuel Alonso Morón

1. INTRODUCTION.

2. FIRST EXAMPLE: The geometric series and the Banach Fixed Point Theorem.

3. FIRST QUESTION: Can we sum the arithmetic-geometric series in a similar way?

4. FIRST APPROACH: The first remainder.

5. SECOND APPROACH: A generalized Banach fixed point theorem.

6. THE $T(f \mid g)$ GROUP.

7. CONCLUSIONS

PRELIMINARIES

$$\mathbb{K}, \qquad \mathbb{K}[[x]]$$

$$g = \sum_{n \geq 0} g_n x^n, \qquad \omega(g)$$

$$(\mathbb{K}[[x]], d)$$

$$d(f,g) = \frac{1}{2^{\omega(f-g)}}$$

Banach fixed point theorem (BFPT) Let (X,d) be a complete metric space and $f : X \to X$ contractive. Then f has a unique fixed point x_0 and $f^n(x) \to x_0$ for every $x \in X$.

Generalize Banach fixed point theorem (GBFPT) Let (X,d) be a complete metric space. Suppose $\{f_n\}_{n\in\mathbb{N}} : X \longrightarrow X$ is a sequence of contractive maps with the same contraction constant α and suppose that $\{f_n\} \longrightarrow f$ (point to point). Then f is α-contractive and for any point $z \in X$ the sequence $\{f_n \circ \cdots \circ f_1(z)\} \xrightarrow[n\to\infty]{} x_0$, where x_0 is the unique fixed point of f.

FIRST EXAMPLE: The geometric series.

$$\begin{array}{rcl} f : (\mathbb{K}[[x]], d) & \to & (\mathbb{K}[[x]], d) \\ t & \mapsto & xt + 1 \end{array}$$

$$f \text{ is contractive, } d(f(t_1), f(t_2)) \leq \frac{1}{2} d(t_1, t_2)$$

$$f(0) = 1$$

$$f^2(0) = x + 1$$

$$f^3(0) = x^2 + x + 1$$

$$f^4(0) = x^3 + x^2 + x + 1$$

$$f^{n+1}(0) = \sum_{k=0}^{n} x^k$$

$$f(t) = xt + 1 = t \qquad \Rightarrow \qquad t = \frac{1}{1-x}$$

$$\sum_{k=0}^{n} x^k \quad \xrightarrow[n \to \infty]{} \quad \frac{1}{1-x}$$

Can we sum the arithmetic-geometric series $\sum_{k=1}^{\infty} kx^{k-1}$ using the BFPT?.

$$f(t) = g(x)t + h(x),\ x_0\ /\ f^{n+1}(x_0) \neq \sum_{k=0}^{n} (k+1)x^k$$

$f(x_0) = g(x)x_0 + h(x) = 1,$

$f^2(x_0) = g(x) + h(x) = 1 + 2x$

$f^3(x_0) = g(x)(1 + 2x) + h(x) = 1 + 2x + 3x^2$

$x_0 = -\frac{1}{3}$ and $f(x) = \frac{3}{2}xt + 1 + \frac{1}{2}x$ but

$f^4(-\frac{1}{3}) \neq 1 + 2x + 3x^2 + 4x^3.$

FIRST APPROACH: The first remainder.

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$t = \frac{1}{(1-x)^2} \qquad \Rightarrow \qquad t = 1 + (2x - x^2)t$$

$$f(t) = 1 + (2x - x^2)t$$

$f(0) = 1$

$f^2(0) = 1 + 2x \mathbf{- x^2}$

$f^3(0) = 1 + 2x + 3x^2 \mathbf{- 4x^3 + x^4}$

$f^4(0) = 1 + 2x + 3x^2 + 4x^3 \mathbf{- 11x^4 + 6x^5 - x^6}$

$f^5(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \mathbf{- 26x^5 + 23x^6 - 8x^7}$

$\mathbf{+ x^8}$

$f^6(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 \mathbf{- 57x^6 +}$

$\mathbf{+ 72x^7 - 39x^8 + 10x^9 - x^{10}}$

$$\begin{pmatrix} -1 & & & & & & \\ -4 & 1 & & & & & \\ -11 & 6 & -1 & & & & \\ -26 & 23 & -8 & 1 & & & \\ -57 & 72 & -39 & 10 & -1 & & \\ -120 & 201 & -150 & 59 & -12 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. $a_{n,k} = 2a_{n-1,k} - a_{n-1,k-1}$.

2. Eulerian numbers.

3. Triangular numbers.

4. $\sum_{k=1}^{i-1} a_{kj} + \sum_{k=j+1}^{n} a_{ik} = 0$.

5. $a_{n,j} = n+j-1+\sum_{k=1}^{j-1}(-1)^k\binom{n+j-1-k}{n+j-2k}2^{n+j-2k}$.

SECOND APPROACH: A generalized Banach fixed point theorem GBFPT

Equicontractive sequence: $h_0(t) = xt$, $h_1(t) = xt+x$, $h_2(t) = xt+x+x^2$, $h_3(t) = xt+x+x^2+x^3$,

$$h_m(t) = xt + x\sum_{k=0}^{m-1} x^k$$

$$\{h_m\} \longrightarrow h(t) = xt + \frac{x}{1-x}$$

$$h_0(0) = 0$$

$$h_1(h_0(0)) = x$$

$$h_2(h_1(h_0(0))) = x + 2x^2$$

$$h_3(h_2(h_1(h_0(0)))) = x + 2x^2 + 3x^3$$

Using **GBFPT** and $xt + \frac{x}{1-x} = t \Rightarrow t = \frac{x}{(1-x)^2}$

$$(h_m \circ \cdots \circ h_0)(0) \longrightarrow \frac{x}{(1-x)^2}$$

$$
\begin{array}{ccccccccc}
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
1 & 2 & 1 & & & & & & \\
1 & 3 & 3 & 1 & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \binom{n}{5} & \binom{n}{6} & \cdots & \binom{n}{n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{1-x} & \frac{x}{(1-x)^2} & \frac{x^2}{(1-x)^3} & \frac{x^3}{(1-x)^4} & \frac{x^4}{(1-x)^5} & \frac{x^5}{(1-x)^6} & \frac{x^6}{(1-x)^7} & \cdots & \frac{x^{n-1}}{(1-x)^n}
\end{array}
$$

$$h_{m,1}(t) = xt + x \sum_{k=0}^{m-1} x^k = xt + xT_{m-1,1}$$

$$h_{m,2}(t) = xt + x \sum_{k=0}^{m-1} kx^k = xt + xT_{m-1,2}$$

$$h_2(t) = xt + x\frac{x}{(1-x)^2} \Rightarrow t = \frac{x^2}{(1-x)^3}$$

$$
\begin{aligned}
&h_{0,2}(0) = h_{1,2}(h_{0,2}(0)) = 0, \\
&h_{2,2}(h_{1,2}(h_{0,2}(0))) = x^2, \\
&h_{3,2}(h_{2,2}(h_{1,2}(h_{0,2}(0)))) = x^2 + 3x^3, \\
&h_{4,2}(h_{3,2}(h_{2,2}(h_{1,2}(h_{0,2}(0))))) = x^2 + 3x^3 + 6x^4,
\end{aligned}
$$

$$(h_{m,2} \cdots h_{0,2})(0) = \sum_{k=0}^{m} \binom{k}{2} x^k \longrightarrow \frac{x^2}{(1-x)^3}.$$

Proposition 1. For $n \geq 2$, the n-column in Pascal's triangle is obtained from the (n-1)-column applying the crossed iterations in **GBFPT** to the sequence $\{h_{m,n}\}_{m\in\mathbb{N}}$ where

$$h_{m,n}(t) = xt + xT_{m-1,n-1}$$

being $T_{m-1,n-1}$ the (m-1)-Taylor polynomial of the (n-1)-column.

THE $T(f \mid g)$ GROUP.

$$f = \sum_{n \geq 0} f_n x^n, \;\; g = \sum_{n \geq 0} g_n x^n, \;\; \frac{f}{g} = \sum_{n \geq 0} d_n x^n$$

$$\begin{array}{c|ccccc} f_0 & & & & \\ f_1 & c_{1,1} & & & \\ f_2 & c_{1,1} & c_{1,2} & & \\ f_3 & c_{2,1} & c_{2,2} & c_{2,3} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ f & \frac{f}{g} & \frac{xf}{g^2} & \frac{x^2 f}{g^3} & \cdots \end{array}$$

$$h_{m,n}(t) = T_m \left(\frac{g_0 - g}{g_0} \right) t + x T_{m-1,n-1} \left(\frac{x^{n-2} f}{g_0 g^{n-1}} \right)$$

$$h_n(t) = \left(\frac{g_0 - g}{g_0} \right) t + x \left(\frac{x^{n-2} f}{g_0 g^{n-1}} \right) \;\Rightarrow\; t = \frac{x^{n-1} f}{g^n}$$

$$d_0 = \frac{f_0}{g_0}$$

$$d_n = -\frac{g_1}{g_0} d_{n-1} - \frac{g_2}{g_0} d_{n-2} \cdots - \frac{g_n}{g_0} d_0 + \frac{f_n}{g_0}$$

Algorithm for $T(f \mid g) = (c_{i,j})$

$$\begin{array}{c|cccccc}
f_0 & & & & & & \\
f_1 & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & \cdots \\
f_2 & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & \cdots \\
f_3 & c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
f_i & c_{i1} & c_{i2} & c_{i3} & c_{i4} & c_{i5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

with $c_{i,j} = 0$ if $j > i$ and for $i \geq j$. For $j \geq 2$:

$$c_{i,j} = -\frac{g_1}{g_0} c_{i-1,j} - \frac{g_2}{g_0} c_{i-2,j} \cdots - \frac{g_{i-1}}{g_0} c_{1,j} + \frac{c_{i-1,j-1}}{g_0}$$

$$c_{i,1} = -\frac{g_1}{g_0} c_{i-1,1} - \frac{g_2}{g_0} c_{i-2,1} \cdots - \frac{g_{i-1}}{g_0} c_{1,1} + \frac{f_{i-1}}{g_0}$$

$$
\begin{array}{rccc}
T(f \mid g): & (\mathbb{K}[[x]], d) & \rightarrow & (\mathbb{K}[[x]], d) \\
& h & \mapsto & T(f \mid g)(h) = \frac{f}{g} h\left(\frac{x}{g}\right)
\end{array}
$$

$$
T(f_1 \mid g_1) T(f_2 \mid g_2) = T\left(f_1 f_2\left(\frac{x}{g_1}\right) \Big| g_1 g_2\left(\frac{x}{g_1}\right)\right)
$$

$$
(T(f \mid g))^{-1} = T\left(\frac{1}{f(k^{-1})} \Big| \frac{1}{g(k^{-1})}\right), \qquad k = \frac{x}{g}
$$

Shapiro and $T(f \mid g)$ Notations

Name	**(d(t),th(t))**	**T(f|g)**
Identity	$(1, t)$	$T(1 \mid 1)$
Pascal	$\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$	$T(1 \mid 1-t)$
Remainders	$\left(\frac{1}{Q(t)}\frac{-c}{a+bt}, \frac{-ct}{a+bt}\right)$	$T\left(\frac{1}{Q} \mid \frac{a+bt}{-c}\right)$
Appel subgroup element	(d(t),t)	$T(d \mid 1)$
Associated subgroup element	(1,th(t))	$T\left(\frac{1}{h} \mid \frac{1}{h}\right)$
Bell subgroup element	(d(t),td(t))	$T\left(1 \mid \frac{1}{d}\right)$

Sprugnoli and $T(f \mid g)$ Notations

Name	**(d(t),h(t))**	$T(f \mid g)$
Pascal	$\left(\frac{1}{1-t}, \frac{1}{1-t}\right)$	$T(1 \mid 1-t)$
Appel	(d(t),1)	$T(d \mid 1)$
Associated	(1,h(t))	$T\left(\frac{1}{h} \mid \frac{1}{h}\right)$
Bell	(d(t),d(t))	$T\left(1 \mid \frac{1}{d}\right)$
Stirling 1	$\left(1, \frac{1}{t} \ln \frac{1}{1-t}\right)$	$T\left(\frac{-t}{\ln(1-t)} \mid \frac{-t}{\ln(1-t)}\right)$
Stirling 2	$\left(1, \frac{e^t-1}{t}\right)$	$T\left(\frac{t}{e^t-1} \mid \frac{t}{e^t-1}\right)$

Fundamental equality

$$\boxed{T(f|g) = T(f|1)T(1|g)}$$

$$\left(\frac{f(t)}{g(t)}, \frac{t}{g(t)}\right) = (f(t), t)\left(\frac{1}{g(t)}, \frac{t}{g(t)}\right)$$

$$\left(\frac{f(t)}{g(t)}, \frac{1}{g(t)}\right) = (f(t), 1)\left(\frac{1}{g(t)}, \frac{1}{g(t)}\right)$$

$$T^{-1}(1 \mid g) = T(1 \mid A)$$

$$T^{-1}(f \mid g) = T\left(\frac{g_0}{f_0}(A - xZ)\Big| A\right)$$

$$\begin{pmatrix} -1 & & & & & & \\ -4 & 1 & & & & & \\ -11 & 6 & -1 & & & & \\ -26 & 23 & -8 & 1 & & & \\ -57 & 72 & -39 & 10 & -1 & & \\ -120 & 201 & -150 & 59 & -12 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = T\left(\frac{1}{(1-x)^2} \mid 2x-1\right)$$

$$Q(x) = a + bx + cx^2, a \neq 0$$

$$T\left(\frac{1}{Q} \mid \frac{a+bx}{-c}\right)$$

Theorem

$$T\left(\frac{1}{Q} \mid \frac{a+bx}{-c}\right) = T\left(\frac{1}{Q} \mid 1-x\right) T^{-1}(1 \mid 1-x) T\left(1 \mid \frac{a+bx}{-c}\right)$$

or equivalently

$$T\left(\frac{1}{Q} \mid 1-x\right) = T\left(\frac{1}{Q} \mid \frac{a+bx}{-c}\right) T^{-1}\left(1 \mid \frac{a+bx}{-c}\right) T(1 \mid 1-x)$$

or equivalently

$$T(1 \mid 1-x) = T\left(1 \mid \frac{a+bx}{-c}\right) T^{-1}\left(\frac{1}{Q} \mid \frac{a+bx}{-c}\right) T\left(\frac{1}{Q} \mid 1-x\right)$$

$$T(1 \mid a + bx)$$

p_n associated to $T(f \mid g) = (c_{n,k})_{n,k \in \mathbb{N}}$

$$p_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$$

$$T(f \mid ag + bx)$$

$$q_n(t) = \frac{1}{a} \sum_{k=0}^{n} c_{n,k} t^k \text{ with } t = \frac{x-b}{a}$$

$$T(f \mid ag + bx) = T(f|g)T(1|a + bx)$$

Conclusions:

- By means of Banach's fixed point theorem we run into the Riordan group.

- In some sense, we reinforce the idea of the ubiquity of Pascal triangle.

- The algorithm of dividing series allow us to get a general algorithm to construct all Riordan arrays avoiding both usual A and Z sequences.

References:

Ultrametrics, Banach's fixed point theorem and the Riordan group. Discr. Appl. Math. (In press.) (2007) doi:10.1016/j.dam.2007.10.026. A. L. and M. A. Morón.

Riordan matrices. in the reciprocation of quadratic polynomials (Submitted.) (2008) A. L. and M. A. Morón.