Polynomial sequences associated to the stochastic subgroup of the Riordan group.

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- R.P. Boas, R.C. Buck. *Polynomial expansions of analytic functions.* Springer-Verlag (1964.)

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- T-X. He, L.C. Hsu, P.J-S. Shiue. *The Sheffer group and the Riordan group.* Discrete Applied Mathematics. 155 (2007) 1895-1909.

- R.P. Boas, R.C. Buck. *Polynomial expansions of analytic functions.* Springer-Verlag (1964.)
- T-X. He, L.C. Hsu, P.J-S. Shiue. *The Sheffer group and the Riordan group.* Discrete Applied Mathematics. 155 (2007) 1895-1909.
- A. L. and Manuel A. Morón. *Recurrence relations for poly nomial sequences via Riordan matrices.* Linear Algebra and its Applications. 433 (2010) 1422-1446.

Outline

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1. The Riordan group.
2. Classical families of polynomials.
3. The stochastic subgroup and its polynomial family associated.

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In the previous formula we understand that $\frac{1}{2^\infty} = 0$.

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$(\mathcal{R}, \cdot)$ is the Riordan group

$$
\begin{array}{cccc}
T(f\mid g): & (\mathbb{K}[[x]],d) & \to & (\mathbb{K}[[x]],d) \\
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$$\frac{f}{g} h\left(\frac{x}{g}\right) \equiv \frac{f(x)}{g(x)} h\left(\frac{x}{g(x)}\right)$$

Pascal's triangle.

$$P = T(1 \mid 1-x) = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad P_n(x) = F_n(2x)$$

Polynomials of Riordan type

Definition. $(p_n(x))_{n\in\mathbb{N}}$ is a *polynomial sequence of Riordan type* if $(d_{n,k})_{n,k\in\mathbb{N}}$ is an element of the Riordan group.

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Umbral Composition

$$f=\sum_{n\geq 0} f_n x^n,\ g=\sum_{n\geq 0} g_n x^n,\ l=\sum_{n\geq 0} l_n x^n,\ m=\sum_{n\geq 0} m_n x^n$$

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$$r_n(x)=\sum_{k=0}^{n} p_{n,k}q_k(x)$$

Recurrence of polynomials family.

Main theorem

Let $D = (d_{n,j})_{n,j \in \mathbb{N}}$ be an i.l.t.m. D is a Riordan matrix iff $\exists$ (f_n) and (g_n), $g_0 \neq 0$

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Let $D=(d_{n,j})_{n,j\in\mathbb{N}}$ be an i.l.t.m. D is a Riordan matrix iff $\exists$ (f_n) and (g_n), $g_0\neq 0$

$$p_n(x)=\left(\frac{x-g_1}{g_0}\right)p_{n-1}(x)-\frac{g_2}{g_0}p_{n-2}(x)\cdots-\frac{g_n}{g_0}p_0(x)+\frac{f_n}{g_0}\qquad \forall n\geq 0$$

Moreover $D=T(f\mid g)$ where $f=\sum_{n\geq 0}f_nx^n$ and $g=\sum_{n\geq 0}g_nx^n$.

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Proposition.

Let $f = \sum_{n\geq 0} f_n x^n$, $g = \sum_{n\geq 0} g_n x^n$ be two power series such that $f_0 \neq 0,\ g_0 \neq 0$. Suppose that $(p_n(x))_{n\in\mathbb{N}}$ is the associated polynomial sequence of the Riordan array $T(f \mid g)$, then

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(a) If $(q_n(x))_{n\in\mathbb{N}}$ is the associated sequence to $T(fg \mid g)$ we obtain

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(b) If $(r_n(x))_{n\in\mathbb{N}}$ is the associated polynomial sequence to $T\left(\frac{f}{g}\Big|g\right)$ then

$$r_{n-1}(x) = \frac{p_n(x) - p_n(0)}{x} \quad \text{for} \quad n \geq 1$$

Corollary

Suppose $g = \sum_{n\geq 0} g_n x^n$ with $g_0 \neq 0$. Let $(p_n(x))_{n\in\mathbb{N}}$ be the pol. seq. ass. to $T(1 \mid g)$
$(q_n(x))_{n\in\mathbb{N}}$ that associated to $T(g \mid g)$.

$$q_n(x) = xp_{n-1}(x) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad q_0(x) = 1$$

Morgan-Voyce polynomials.

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 3 & 4 & 1 & & & \\ 4 & 10 & 6 & 1 & & \\ 5 & 20 & 21 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$B_0(x) = 1,\ B_1(x) = 2+x,\ B_2(x) = 3+4x+x^2, \cdots$$

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$$B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x)$$

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x)$$

Morgan-Voyce polynomials.

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 3 & 4 & 1 & & & \\ 4 & 10 & 6 & 1 & & \\ 5 & 20 & 21 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 5 & 1 & & \\ 1 & 10 & 15 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$B_0(x) = 1,\ B_1(x) = 2+x,\ B_2(x) = 3+4x+x^2, \cdots \qquad b_0(x) = 1,\ b_1(x) = 1+x,\ b_2(x) = 1+3x+x^2, \cdots$$

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$$T(1-x|1)\,T(1|(1-x)^2) = T(1-x|(1-x)^2)$$

$$T(1-x|1)\,T(1-x|(1-x)^2) = T((1-x)^2|(1-x)^2)$$

Proposition.

Let $T(f \mid g)$ and $T(l \mid m)$ be two element of the Riordan group. Suppose that $(p_n(x))$ and $(q_n(x))$ are the corresponding associated families of polynomials.

$$T(l \mid m) = T(\gamma \mid \alpha + \beta x) T(f \mid g) T(c \mid a + bx)$$

where $\alpha, \gamma, a, c \neq 0$. Then

$$q_n(x) = \frac{\gamma c}{\alpha a} \left(\sum_{k=0}^{n} \binom{n}{k} \left(-\frac{\beta}{\alpha} \right)^{n-k} \frac{1}{\alpha^k} p_k \left(\frac{x-b}{a} \right) \right)$$

Example: Fibonacci and Pell polynomials

$T(1 \mid 1 - x^2)$ Fibonacci

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Fibonacci and Chebychev of second kind polynomials

$$T\left(\frac{1}{2}\middle|\frac{1}{2}(1+x^2)\right)=T\left(\frac{1}{2}\middle|-i\right)T\left(1\middle|1-x^2\right)T\left(1\middle|\frac{i}{2}\right)$$

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$$U_n(x) = i^n F_n(-2ix)$$

Fermat and Chebychev of second kind polynomials

Fermat polynomials are $\mathcal{F}_0(x) = 1$, $\mathcal{F}_1(x) = 3x$ and for $n \geq 2$

$$\mathcal{F}_n(x) = 3x\mathcal{F}_{n-1}(x) - 2\mathcal{F}_{n-2}(x)$$

$$T\left(\frac{1}{3}\middle|\frac{1}{3}+\frac{2}{3}x^2\right) = \begin{pmatrix} 1 & & & & & & & \\ 0 & 3 & & & & & & \\ -2 & 0 & 9 & & & & & \\ 0 & -12 & 0 & 27 & & & & \\ 4 & 0 & -54 & 0 & 81 & & & \\ 0 & 36 & 0 & -216 & 0 & 243 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$\mathcal{F}_n(x) = (\sqrt{2})^n U_n\left(\frac{3x}{2\sqrt{2}}\right)$$

Generalized Appell polynomials. (Boas-Buck, 1964)

$(s_n(x))$ is a *family of generalized Appell polynomials* iff $\exists$ $f, g, h \in \mathbb{K}[[x]]$, with $f_0, g_0 \neq 0$, and $h_n \neq 0$ for all n such that

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Sheffer polynomials.

$$T(f \mid g)(e^{tx}) = \sum_{n \geq 0} S_n(t) x^n$$

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WARNING:

Often called a Sheffer sequence to the sequence $(n!S_n(x))_{n\in\mathbb{N}}$ where $(S_n(x))_{n\in\mathbb{N}}$ is our Sheffer sequence.

Pidduck and Mittag-Leffler polynomials.

Consider $(\mathcal{P}_n(x))$

$$\sum_{n\geq 0} \mathcal{P}_n(t)x^n = T\left(\frac{x}{(1-x)\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)(e^{tx})$$

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If $(M_n(x))$ is given by the formula:

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$$\mathcal{P}_n(x) = \sum_{k=0}^{n} M_k(x) \quad \Leftrightarrow \quad \widetilde{P}_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-k)!\widetilde{M}_k(x)$$

Example: Laguerre polynomials.

$$T(-1 \mid x-1)(e^{tx}) = T(1 \mid 1-x)T(-1 \mid -1)(e^{tx}) =$$

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Example: Hermite polynomials.

$$\sum_{n\geq 0} H_n(t)x^n = T\left(\frac{1}{2e^{x^2}}\middle|\frac{1}{2}\right)(e^{tx}) =$$

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$$H_n(x) = xH_{n-1}(x) \star \widehat{h}(x) + f_n$$

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$$\widehat{h}(x) = \sum_{n\geq 1} \frac{2}{n}x^n = -2\log(1-x)$$

$$f_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}, & \text{if } n \text{ is even.} \end{cases}$$

Subgroups and families of polynomials.

Brenke polynomials and Appell subgroup.

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$$T(f \mid 1)(h(tx)) = \sum_{n\geq 0} B_n(t)x^n$$

$$T(f \mid 1)(e^{tx}) = \sum_{n\geq 0} A_n(t)x^n$$

Subgroups and families of polynomials.

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$$T(f \mid 1)\left(\frac{1}{1-tx}\right) = \sum_{n\geq 0} T_n^*(t)x^n$$

Faber polynomials.

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- Gi-Sang Cheon, Hana Kim and Louis W. Shapiro, 'An algebraic structure for Faber polynomials.' (2010) relate the Faber polynomials with the Hitting-time subgroup of the Riordan group.

The Stochastic subgroup.

The stabilizer of geometric progression.

$$T\,(f \mid g)\left(\frac{1}{1-x}\right)=\frac{1}{1-x} \quad\Leftrightarrow\quad f=\frac{g-x}{1-x}$$

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The g-stochastic triangle and its polynomial sequence.

$$\sum_{n\geq 0} p_n(t)x^n = T\left(\frac{g-x}{1-x}\Big| g\right)\left(\frac{1}{1-xt}\right)=\frac{g-x}{(1-x)(g-xt)}$$

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The associated family of polynomials, $(p_n(x))_{n\in\mathbb{N}}$, of any element of the Stochastic subgroup holds

$$p_n(1)=1$$

Proposition: Characterization of stochastic families.

$(p_n(x))_{n\in\mathbb{N}}$ is a stochastic family if and only if there are $(g_n)_{n\in\mathbb{N}}$, with $g_0 \neq 0$ such that

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right) p_{n-1}(x) - \frac{g_2}{g_0} p_{n-2}(x) \cdots - \frac{g_n}{g_0} p_0(x) + \frac{1}{g_0}\left(\sum_{k=0}^{n} g_k - 1\right)$$

$\forall n \geq 0.$

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$\forall n \geq 0.$

If $g = 1$, the element of the stochastic group is $T(1 \mid 1)$, and then, the polynomial family is $p_n(x) = x^n$.

$(1-x)$-stochastic triangle

$$T\left(\frac{1-2x}{1-x}\middle|\, 1-x\right)$$

$(1-x)$-stochastic triangle

$$T\left(\frac{1-2x}{1-x}\middle| 1-x\right)$$

$$\begin{pmatrix}
1 \\
0 & 1 \\
-1 & 1 & 1 \\
-2 & 0 & 2 & 1 \\
-3 & -2 & 2 & 3 & 1 \\
-4 & -5 & 0 & 5 & 4 & 1 \\
-5 & -9 & -5 & 5 & 9 & 5 & 1 \\
-6 & -14 & -14 & 0 & 14 & 14 & 6 & 1 \\
-7 & -20 & -28 & -14 & 14 & 28 & 20 & 7 & 1 \\
-8 & -27 & -48 & -42 & 0 & 42 & 2 & 48 & 27 & 8 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

The family of associated polynomials is given by

$$\sum_{n\geq 0} p_n(t)x^n = T\left(\frac{1-2x}{1-x}\Big|1-x\right)\left(\frac{1}{1-xt}\right) = \frac{1-2x}{(1-x)(1-x-xt)}$$

$$p_n(x) = (x+1)p_{n-1}(x) + 1$$

with $p_0(x) = 1$. Or equivalently

$$p_n(x) = (1+x)^n - \sum_{k=0}^{n-1}(1+x)^k$$

$\dfrac{1}{1-x}$-stochastic triangle

$$T\left(\frac{1-x+x^2}{(1-x)^2}\,\middle|\,\frac{1}{1-x}\right)$$

$\dfrac{1}{1-x}$-stochastic triangle

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$$\begin{pmatrix}
1 \\
0 & 1 \\
1 & -1 & 1 \\
1 & 1 & -2 & 1 \\
1 & 0 & 2 & -3 & 1 \\
1 & 0 & -1 & 4 & -4 & 1 \\
1 & 0 & 0 & -3 & 7 & -5 & 1 \\
1 & 0 & 0 & 1 & -7 & 11 & -6 & 1 \\
1 & 0 & 0 & 0 & 4 & -14 & 16 & -7 & 1 \\
1 & 0 & 0 & 0 & -1 & 11 & -25 & 22 & -8 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

The family of associated polynomials is

$$\sum_{n\geq 0} p_n(t)x^n = T\left(\frac{1-x+x^2}{(1-x)^2}\Big|\frac{1}{1-x}\right)\left(\frac{1}{1-xt}\right)$$

$$= \frac{1-x+x^2}{(1-x)(1-(1-x)xt)}$$

$$p_n(x) = xp_{n-1}(x) - \sum_{k=0}^{n-1} p_k(x) + n$$

with $p_0(x) = 1$.