Groups of finite Riordan matrices.

Riordan Arrays and Related Topics.

Ana Luzón

Universidad Politécnica de Madrid. España.

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Outline

1. Finite Riordan matrices.

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2. Reflections on finite Riordan matrices.

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2. Reflections on finite Riordan matrices.
3. Reflections on bi-infinite Riordan matrices.

Finite Riordan Matrices.

Definition.

$D = (d_{i,j})_{i,j=0,1,\cdots n}$, $n \geq 1$. D lower triangular.
D is a **Finite Riordan Matrix** $\Leftrightarrow$
$d_{0,0} \neq 0$, and $g_0, g_1, \cdots, g_{n-1}$ in $\mathbb{K}$ with $g_0 \neq 0$

$$d_{i,j} = \sum_{k=0}^{i-j} g_k d_{i+1-k,j+1} \qquad i,j = 0 \cdots n-1 \tag{1}$$

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Definition (continuation)

Moreover if $g(x) = \sum_{k=0}^{n-1} g_k x^k$ and $A(x) = \sum_{k=0}^{n-1} a_k x^k$, then

$$[x^{k-1}]\frac{1}{A(x)} = \frac{1}{k}[x^{k-1}]g^k(x) \text{ for } k = 1 \cdots n; \qquad (3)$$

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Example

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & \boxed{2} & 1 & \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

A(x)=1+x, g(x)=1-x

Finite Riordan matrices.

$$D = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -2 & 1 & & \\ 0 & -1 & -3 & 1 & \\ 5 & 2 & 0 & -4 & 1 \end{pmatrix}$$

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$$\mathcal{R} = \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$$

Groups of finite Riordan matrices.

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$$\mathcal{R}_3 = \left\{ \begin{pmatrix} r_0 & 0 & 0 & 0 \\ d_0 & r_0 r & 0 & 0 \\ \alpha & r(d_0+d) & r_0 r^2 & 0 \\ \beta & \gamma & r^2(d_0+2d) & r_0 r^3 \end{pmatrix} \;\middle|\; \begin{matrix} r_0, r \neq 0 \\ d_0, d, \alpha, \beta, \gamma \in \mathbb{K} \end{matrix} \right\}$$

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Theorem: The reflected Riordan matrix.

$D \in \mathcal{R}_n$, $D = (d_{i,j})_{i,j=0,1\cdots,n}$, then $D^R = (c_{i,j})_{i,j=0,1\cdots,n}$ with $c_{i,j} = d_{n-j,n-i}$ $D^R \in \mathcal{R}_n$.

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$$Q_4\left(\begin{pmatrix} 1 & & & & & \\ 5 & 1 & & & & \\ 10 & 4 & 1 & & & \\ 10 & 6 & 3 & 1 & & \\ 5 & 4 & 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & & & & \\ 4 & 1 & & & \\ 6 & 3 & 1 & & \\ 4 & 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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$$(Q_4(D^R))^R = P_4(D)$$

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Theorem

Let $V(d,g)$ be any Riordan matrix. Then

(i) The sequence $(V_n^R(dg^{n+1}, g))_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, is the **complementary** Riordan matrix of $V(d,g)$ denoted by $V^{\perp}(d,g)$.

Proposition

$(D_n)_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$ if and only if $D_n^R = Q_n(D_{n+1}^R)$ for all $n \in \mathbb{N}$.

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(ii) The sequence $(V_n^R(dg^n, g))_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, is the **dual** Riordan matrix of $V(d, g)$ denoted by $V^{\diamondsuit}(d, g)$.

The Pascal's Triangle and its complementary*.

$$\begin{pmatrix} \ddots & & & & & & & & & \\ \cdots & 1 & & & & & & & & \\ \cdots & -3 & 1 & & & & & & & \\ \cdots & 3 & -2 & 1 & & & & & & \\ \cdots & -1 & 1 & -1 & 1 & & & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 2 & 1 & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$\begin{pmatrix} \ddots & & & & & & & & & & \\ \cdots & 1 & & & & & & & & & \\ \cdots & -3 & 1 & & & & & & & & \\ \cdots & 3 & -2 & 1 & & & & & & & \\ \cdots & -1 & 1 & -1 & 1 & & & & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & & & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 2 & 1 & & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & & \\ \cdots & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Relations between V_n^R, $V_n^\perp$, $V_n^\diamondsuit$

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The isomorphism Φ

There is an unique isomorphism $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\Pi_n \circ \Phi = Q_n \circ \Pi_{n+1}$ for every $n \in \mathbb{N}$.

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The bi-infinite representation.

If $G_n = \mathcal{R}$ and $\Psi_n = \Phi$ for all n,

$$\mathcal{R} = \varprojlim \{(G_n)_{n\in\mathbb{N}}, (\Psi_n)_{n\in\mathbb{N}}\}$$

The bi-infinite Riordan matrix.

$$B(d,g)=\begin{pmatrix} \ddots & & & & & & & \\ \cdots & d_{-3,-3} & & & & & & \\ \cdots & d_{-2,-3} & d_{-2,-2} & & & & & \\ \cdots & d_{-1,-3} & d_{-1,-2} & d_{-1,-1} & & & & \\ \cdots & d_{0,-3} & d_{0,-2} & d_{0,-1} & d_{0,0} & & & \\ \cdots & d_{1,-3} & d_{1,-2} & d_{1,-1} & d_{1,0} & d_{1,1} & & \\ \cdots & d_{2,-3} & d_{2,-2} & d_{2,-1} & d_{2,0} & d_{2,1} & d_{2,2} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The bi-infinite Riordan matrix from finite matrices.

Given $B(d,g)$ we define $\gamma_n(B(d,g)) = V_{2n}(dg^n, g)$ for $n \geq 0$.

$$\cdots, \begin{pmatrix} d_{-2,-2} & & & & \\ d_{-1,-2} & d_{-1,-1} & & & \\ d_{0,-2} & d_{0,-1} & d_{0,0} & & \\ d_{1,-2} & d_{1,-1} & d_{1,0} & d_{1,1} & \\ d_{2,-2} & d_{2,-1} & d_{2,0} & d_{2,1} & d_{2,2} \end{pmatrix}, \begin{pmatrix} d_{-1,-1} & & \\ d_{0,-1} & d_{0,0} & \\ d_{1,-1} & d_{1,0} & d_{1,1} \end{pmatrix}, \begin{pmatrix} d_{0,0} \end{pmatrix}$$

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Analogously, given $B(d,g)$ we define $\delta_n(B(d,g)) = V_{2n+1}(dg^n, g)$ for $n \geq 0$.

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The bi-infinite Riordan matrix from finite matrices.

(i)

$$\mathcal{R} = \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$$

where

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Reflections in bi-infinite matrices.

$$\cdots,\begin{pmatrix} d_{2,2} & & & & \\ d_{2,1} & d_{1,1} & & & \\ d_{2,0} & d_{1,0} & d_{0,0} & & \\ d_{2,-1} & d_{1,-1} & d_{0,-1} & d_{-1,-1} & \\ d_{2,-2} & d_{1,-2} & d_{0,-2} & d_{-1,-2} & d_{-2,-2} \end{pmatrix},\begin{pmatrix} d_{1,1} & & \\ d_{1,0} & d_{0,0} & \\ d_{1,-1} & d_{0,-1} & d_{-1,-1} \end{pmatrix},\ \begin{pmatrix} d_{0,0} \end{pmatrix}$$

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Reflections in bi-infinite matrices.

The odd reflection.

Let $B(d,g) = (D_{2n})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$ then

$$(D^R_{2n})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$$

and it represents a bi-infinite Riordan matrix,

$$(D^R_{2n})_{n\geq 0} = B^{R_o}(d,g).$$

Moreover

$$B^{R_o}(d,g) = B\left(A\left(\frac{x}{A}\right)' d\left(\frac{x}{A}\right), A\right)$$

The even reflection.

Let $B(d,g) = (D_{2n+1})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$ then

$$(D^R_{2n+1})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$$

and it represents a bi-infinite Riordan matrix,

$$(D^R_{2n+1})_{n\geq 0} = B^{R_e}(d,g).$$

Moreover

$$B^{R_e}(d,g) = B\left((A - xA')d\left(\frac{x}{A}\right), A\right)$$

Remark:

$$B^{R_o}(d,g) = B^{\diamondsuit}(d,g),$$

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Even and odd symmetric matrices.

$D^{R_e} = D$, D is even-symmetric.

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Even and odd symmetric matrices.

$D^{R_e} = D$, D is even-symmetric.
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Even and odd symmetric, dual and complementary matrices.

(i) $V^{\perp}(d,g) = V(d,g)$ if and only if $B(dg,g)$ is an even-symmetric matrix.

(ii) $V^{\diamondsuit}(d,g) = V(d,g)$ if and only if $B(d,g)$ is an odd-symmetric matrix.

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Self reflection and Toeplitz matrix

Let $D_m = (d_{i,j}) \in \mathcal{R}_m$ be such that $D_m = D_m^R$ with $m \geq 1$.

(a) If m is odd then D_m is a Toeplitz matrix.

(b) If m is even and $d_{0,0} = d_{1,1}$, then D_m is a Toeplitz matrix.

Theorem $D^{\perp} = D$

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Corollary

Let $f, \omega \in \mathbb{K}[[x]]$ with $f(0) \neq 0$, $\omega(0) = 0$ and $\omega'(0) \neq 0$ where $\omega'(x)$ is the usual derivative of ω in $\mathbb{K}[[x]]$. The solutions of

$$\begin{cases} x^2 f(\omega(x))\omega'(x) = f(x)\omega^2(x), \\ \omega(\omega(x)) = x \end{cases} \tag{4}$$

in $\mathbb{K}[[x]]$ are just $\omega(x) = x$ and f arbitrary with $f(0) \neq 0$.

Theorem $D^{\diamondsuit} = D$

For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, The Riordan matrix $D = V(d, g)$ such that $D^{\diamondsuit} = D$ have as solutions

$$A(x) = g(x), \qquad d(x) = \lambda \sqrt{g(x) \left(\frac{x}{g(x)} \right)'} e^{\phi\left(x, \frac{x}{g(x)}\right)}$$

with $\lambda \in \mathbb{K}^*$ and $\phi(x, z)$ is a symmetric bivariate power series with $\phi(0, 0) = 0$.

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with $\lambda \in \mathbb{K}^*$ and $\phi(x, z)$ is a symmetric bivariate power series with $\phi(0, 0) = 0$.
If in addition $g(0) = 1$, then $V(d, g)$ is a Toeplitz matrix.

A family of odd symmetric matrices.

In general, if we take $g(x) = \alpha x - 1$, $\phi = 0$ you get that

$$B\left(-\frac{1}{\sqrt{1-\alpha x}}, \alpha x - 1\right) \qquad \alpha \in \mathbb{K}$$

is a family of odd-symmetric bi-infinite matrices and then

$$V\left(-\frac{1}{\sqrt{1-\alpha x}}, \alpha x - 1\right) \qquad \alpha \in \mathbb{K}$$

is a family of self-dual matrices.

An example of $D^{\Diamond} = D$

$$\begin{pmatrix} 1 & & & & & & \\ -5 & -1 & & & & & \\ 15/2 & 3 & 1 & & & & \\ -5/2 & -3/2 & -1 & -1 & & & \\ -5/8 & -1/2 & -1/2 & -1 & 1 & & \\ -3/8 & -3/8 & -1/2 & -3/2 & 3 & -1 & \\ -5/16 & -3/8 & -5/8 & -5/2 & 15/2 & -5 & 1 \end{pmatrix}$$

An example of $D^{\diamondsuit} = D$

$$\begin{pmatrix} 1 & & & & & & \\ -5 & -1 & & & & & \\ 15/2 & 3 & 1 & & & & \\ -5/2 & -3/2 & -1 & -1 & & & \\ -5/8 & -1/2 & -1/2 & -1 & 1 & & \\ -3/8 & -3/8 & -1/2 & -3/2 & 3 & -1 & \\ -5/16 & -3/8 & -5/8 & -5/2 & 15/2 & -5 & 1 \end{pmatrix}$$

$$\gamma_3\left(B\left(-\frac{1}{\sqrt{1-2x}}, 2x-1\right)\right) = B_3\left(-\frac{1}{\sqrt{1-2x}}, 2x-1\right) =$$

$$= V_6\left(-\frac{(2x-1)^3}{\sqrt{1-2x}}, 2x-1\right)$$

Conclusions

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2. The different approximations of the Riordan group by means of finite, infinite or bi-infinite matrices are useful.
3. There are many work to do.