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Classical families of polynomials in Riordan arrays.*

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Introduction: some history

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$t = \frac{1}{(1-x)^2} \qquad \Rightarrow \qquad t = 1 + (2x - x^2)t$$

$$f(t) = 1 + (2x - x^2)t$$

$f(0) = 1$

$f^2(0) = 1 + 2x - \mathbf{x^2}$

$f^3(0) = 1 + 2x + 3x^2 - \mathbf{4x^3} + \mathbf{x^4}$

$f^4(0) = 1 + 2x + 3x^2 + 4x^3 - \mathbf{11x^4} + \mathbf{6x^5} - \mathbf{x^6}$

$f^5(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 -$

$\mathbf{-26x^5} + \mathbf{23x^6} - \mathbf{8x^7} + \mathbf{x^8}$

$f^6(0) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 -$

$\mathbf{-57x^6} + +\mathbf{72x^7} - \mathbf{39x^8} + \mathbf{10x^9} - \mathbf{x^{10}}$

$$p_0(x) = -1$$
$$p_1(x) = -4 + x$$
$$p_2(x) = -11 + 6x - x^2$$
$$p_3(x) = -26 + 23x - 8x^2 + x^3$$
$$p_4(x) = -57 + 72x - 39x^2 + 10x^3 - x^4$$
$$p_5(x) = -120 + 210x - 150x^2 + 59x^3 - 12x^4 + x^5$$

$$\begin{pmatrix} -1 & & & & & & \\ -4 & 1 & & & & & \\ -11 & 6 & -1 & & & & \\ -26 & 23 & -8 & 1 & & & \\ -57 & 72 & -39 & 10 & -1 & & \\ -120 & 201 & -150 & 59 & -12 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$d_{n,k} = 2d_{n-1,k} - d_{n-1,k-1}$$

$$p_n(x) = (2-x)p_{n-1}(x) - (n+1) \qquad n \geq 1$$

Construction of $T(f \mid g)$

$$f = \sum_{n \geq 0} f_n x^n, \quad g = \sum_{n \geq 0} g_n x^n, \quad \text{with} \quad g_0 \neq 0$$

$$T(f \mid g) = (d_{n,k})_{n,k \in \mathbb{N}}$$

$$\left(\begin{array}{c|cccccc} f_0 & & & & & & \\ f_1 & d_{0,0} & d_{0,1} & d_{0,2} & d_{0,3} & d_{0,4} & \cdots \\ f_2 & d_{1,0} & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \cdots \\ f_3 & d_{2,0} & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ f_{n+1} & d_{n,0} & d_{n,1} & d_{n,2} & d_{n,3} & d_{n,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

If $j > n$, $\qquad d_{n,j} = 0$

If $j = 0$, $d_{n,0} = d_n \qquad \frac{f}{g} = \sum_{n \geq 0} d_n x^n$

$$d_{n,0} = -\frac{g_1}{g_0} d_{n-1,0} - \frac{g_2}{g_0} d_{n-2,0} \cdots - \frac{g_n}{g_0} d_{0,0} + \frac{f_n}{g_0}$$

If $j > 0$

$$d_{n,j} = -\frac{g_1}{g_0} d_{n-1,j} - \frac{g_2}{g_0} d_{n-2,j} \cdots - \frac{g_n}{g_0} d_{0,j} + \frac{d_{n-1,j-1}}{g_0}$$

Two formulas

$$T(f \mid g) = T(f \mid 1)T(1 \mid g)$$

$$T^{-1}(1 \mid g) = T(1 \mid A)$$

Conversion formula

$$T(f \mid g) = \left(\frac{f(x)}{g(x)}, \frac{x}{g} \right)$$

$$(d(x), h(x)) = T\left(\frac{xd}{h} \middle| \frac{x}{h} \right)$$

The main recurrence relation is

$$T(f \mid g) = (d_{n,k})_{n,k\in\mathbb{N}}, \qquad p_n(x) = \sum_{k=0}^{n} d_{n,k} x^k$$

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right) p_{n-1}(x) - \frac{g_2}{g_0} p_{n-2}(x) \cdots - \frac{g_n}{g_0} p_0(x) + \frac{f_n}{g_0}$$

Pascal's triangle.

$$\text{Pascal} \equiv T(1 \mid 1-x)$$

$$f = 1 \qquad g = 1-x$$

$$p_n(x) = \left(\frac{x+1}{1}\right) p_{n-1}(x) \quad n \geq 1$$

$$p_n(x) = (x+1)p_{n-1}(x) = (x+1)^n$$

The Fibonacci polynomials

$$F_0(x) = 1,\ F_1(x) = x$$

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{for} \quad n \geq 2$$

$$F_n(x) = \left(\frac{x - g_1}{g_0}\right) F_{n-1}(x) - \frac{g_2}{g_0} F_{n-2}(x) + \frac{f_n}{g_0}$$

$$(f_n)_{n\in\mathbb{N}},\ (g_n)_{n\in\mathbb{N}}$$

$g_0 = 1,\ g_1 = 0,\ g_2 = -1,\ g_n = 0,\ \forall n \geq 3$ and
$f_0 = 1,\ f_n = 0\ \forall n \geq 1,$

$$T\left(1|1 - x^2\right) = (d_{n,k})$$

$$\left(\begin{array}{c|cccccccc}
1 & & & & & & & & \\
0 & 1 & & & & & & & \\
0 & 0 & 1 & & & & & & \\
0 & 1 & 0 & 1 & & & & & \\
0 & 0 & 2 & 0 & 1 & & & & \\
0 & 1 & 0 & 3 & 0 & 1 & & & \\
0 & 0 & 3 & 0 & 4 & 0 & 1 & & \\
0 & 1 & 0 & 6 & 0 & 5 & 0 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$d_{n,k} = d_{n-2,k} + d_{n-1,k-1}$, for $k > 0$, $d_{n,0} = d_{n-2,0}$
for $n \geq 2$, $d_{0,0} = 1$ and $d_{1,0} = 0$.

$$T\left(1|1-x^2\right) = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 1 & 0 & 3 & 0 & 1 & & & \\ 0 & 3 & 0 & 4 & 0 & 1 & & \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$F_0(x) = 1$$
$$F_1(x) = x$$
$$F_2(x) = 1 + x^2$$
$$F_3(x) = 2x + x^3$$
$$F_4(x) = 1 + 3x^2 + x^4$$
$$F_5(x) = 3x + 4x^3 + x^5$$
$$F_6(x) = 1 + 6x^2 + 5x^4 + x^6$$

$$\sum_{n\geq 0} F_n(t)x^n = T\left(1|1-x^2\right)\left(\frac{1}{1-xt}\right) = \frac{1}{1-x^2-xt}$$

The Pell polynomials.

$$P_0(x) = 1 \;\; P_1(x) = 2x$$

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$$

$$\frac{x - g_1}{g_0} = 2x, \;\; \frac{-g_2}{g_0} = 1 \text{ then } g(x) = \frac{1}{2} - \frac{1}{2}x^2 \text{ and } f(x) = \frac{1}{2}.$$

$$T\left(\frac{1}{2}\middle|\frac{1}{2} - \frac{1}{2}x^2\right)$$

$$d_{n,k} = d_{n-2,k} + 2d_{n-1,k-1}, \qquad k > 0$$

$$\left(\begin{array}{c|cccccccc}
\frac{1}{2} & & & & & & & & \\
0 & 1 & & & & & & & \\
0 & 0 & 2 & & & & & & \\
0 & 1 & 0 & 4 & & & & & \\
0 & 0 & 4 & 0 & 8 & & & & \\
0 & 1 & 0 & 12 & 0 & 16 & & & \\
0 & 0 & 6 & 0 & 32 & 0 & 32 & & \\
0 & 1 & 0 & 24 & 0 & 80 & 0 & 64 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$$T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right)=\begin{pmatrix} 1 \\ 0 & 2 \\ 1 & 0 & 4 \\ 0 & 4 & 0 & 8 \\ 1 & 0 & 12 & 0 & 16 \\ 0 & 6 & 0 & 32 & 0 & 32 \\ 1 & 0 & 24 & 0 & 80 & 0 & 64 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{aligned}
P_0(x) &= 1 = F_0(2x) \\
P_1(x) &= 2x = F_1(2x) \\
P_2(x) &= 1+4x^2 = F_2(2x) \\
P_3(x) &= 4x+8x^3 = F_3(2x) \\
P_4(x) &= 1+12x^2+16x^4 = F_4(2x) \\
P_5(x) &= 6x+32x^3+32x^5 = F_5(2x) \\
P_6(x) &= 1+24x^2+80x^4+64x^6 = F_6(2x)
\end{aligned}$$

$$P_n(x) = F_n(2x)$$

$$T\left(\frac{1}{2}\middle|\frac{1}{2}-\frac{1}{2}x^2\right) = T\left(\frac{1}{2}\middle|1\right)T(1|1-x^2)T\left(1\middle|\frac{1}{2}\right)$$

The Morgan-Voyce polynomials.

$$\left(\begin{array}{c|cccccc} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 2 & 1 & & & & \\ 0 & 3 & 4 & 1 & & & \\ 0 & 4 & 10 & 6 & 1 & & \\ 0 & 5 & 20 & 21 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \qquad \left(\begin{array}{c|cccccc} 1 & & & & & & \\ -1 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 1 & 3 & 1 & & & \\ 0 & 1 & 6 & 5 & 1 & & \\ 0 & 1 & 10 & 15 & 7 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

$$B_0(x) = 1 \qquad b_0(x) = 1$$
$$B_1(x) = 2 + x \qquad b_1(x) = 1 + x$$
$$B_2(x) = 3 + 4x + x^2 \qquad b_2(x) = 1 + 3x + x^2$$
$$B_3(x) = 4 + 10x + 6x^2 + x^3 \qquad b_3(x) = 1 + 6x + 5x^2 + x^3$$

In general

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \qquad b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x)$$

$$\sum_{n\geq 0} B_n(t)x^n = T(1|(1-x)^2)\left(\frac{1}{1-xt}\right) = \frac{1}{1-(2+t)x+x^2}$$

$$\sum_{n\geq 0} b_n(t)x^n = T(1-x|(1-x)^2)\left(\frac{1}{1-xt}\right) = \frac{1-x}{1-(2+t)x+x^2}$$

$$B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x)$$

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x)$$

$$B_n(x) - B_{n-1}(x) = b_n(x)$$

$$b_n(x) - b_{n-1}(x) = xB_{n-1}(x)$$

$$T(1-x|1)T(1|(1-x)^2) = T(1-x|(1-x)^2)$$

$$T(1-x|1)T(1-x|(1-x)^2) = T((1-x)^2|(1-x)^2)$$

Definitions and Results

Definition. *The family of polynomials associated to* $A = (a_{n,j})_{n,j\in\mathbb{N}}$ (i.l.t.m.) is $(p_n(x))_{n\in\mathbb{N}}$,

$$p_n(x) = \sum_{j=0}^{n} a_{n,j}x^j, \quad \text{with} \quad n \in \mathbb{N}$$

Main theorem Let $D = (d_{n,j})_{n,j\in\mathbb{N}}$ be an i.l.t.m. D is a Riordan matrix iff $\exists$ (f_n) and (g_n), $g_0 \neq 0$

$$p_n(x) = \left(\frac{x-g_1}{g_0}\right)p_{n-1}(x) - \frac{g_2}{g_0}p_{n-2}(x) \cdots - \frac{g_n}{g_0}p_0(x) + \frac{f_n}{g_0} \qquad \forall n \geq 0$$

Moreover $D = T(f \mid g)$ where $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$.

Umbral Composition

$$f = \sum_{n \geq 0} f_n x^n,\ g = \sum_{n \geq 0} g_n x^n,\ l = \sum_{n \geq 0} l_n x^n,\ m = \sum_{n \geq 0} m_n x^n$$

$$T(f|g) = (p_{n,k})_{n,k \in \mathbb{N}} \qquad T(l|m) = (q_{n,k})_{n,k \in \mathbb{N}}$$

$$(p_n(x))_{n \in \mathbb{N}}, \qquad (q_n(x))_{n \in \mathbb{N}}$$

$$(p_n(x))_{n \in \mathbb{N}} \sharp (q_n(x))_{n \in \mathbb{N}} = (r_n(x))_{n \in \mathbb{N}}$$

$$T(f|g)T(l|m) = T\left(fl\left(\frac{x}{g}\right) \middle| gm\left(\frac{x}{g}\right)\right) = (r_{n,k})_{n,k \in \mathbb{N}}$$

$$r_n(x) = \sum_{k=0}^{n} p_{n,k} q_k(x)$$

$$T(f|g)\left(\frac{1}{1-xt}\right) = \begin{pmatrix} p_{0,0} & & & & & \\ p_{1,0} & p_{1,1} & & & & \\ p_{2,0} & p_{2,1} & p_{2,2} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n} & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \\ \vdots \end{pmatrix} = \sum_{k=0}^{n} p_n(t)x^k$$

$$\sum_{k=0}^{n} p_n(t)x^k = T(f|g)\left(\frac{1}{1-xt}\right) = \frac{f(x)}{g(x)-xt}$$

Definition. $(p_n(x))_{n\in\mathbb{N}}$ is a *polynomial sequence of Riordan type* if $(p_{n,k})_{n,k\in\mathbb{N}}$ is an element of the Riordan group.

$$p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k, \quad \text{with} \quad n \in \mathbb{N}$$

Proposition.

Consider $T(f \mid g)$ and $(p_n(x))$

Consider $T(h \mid 1)T(f \mid g)$ and $(q_n(x))$

Let $h(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_m x^m$ be a m degree polynomial, $h_m \neq 0$.

$$T(h \mid 1)T(f \mid g)$$

$$q_0(x) = h_0 p_0(x)$$

$$q_1(x) = h_1 p_0(x) + h_0 p_1(x)$$

$$\vdots$$

$$q_m(x) = h_m p_{n-m}(x) + \cdots + h_0 p_m(x)$$

If $n \geq m$

$$q_n(x) = h_m p_{n-m}(x) + \cdots + h_0 p_n(x)$$

Example: The Chebyshev polynomials of the first and the second kind.

The Chebyshev polynomials of the second kind:
$U_0(x) = 1$
$U_1(x) = 2x$
$U_2(x) = 4x^2 - 1$
$U_3(x) = 8x^3 - 4x$
$U_4(x) = 16x^4 - 12x^2 + 1$
In general, if $n \geq 2$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

If $U = (u_{n,k})_{n,k \in \mathbb{N}}$

$$U = T\left(\frac{1}{2}\middle|\frac{1}{2} + \frac{1}{2}x^2\right)$$

is a Riordan matrix:

$$\left(\begin{array}{c|cccccc}
\frac{1}{2} & & & & & & \\
0 & 1 & & & & & \\
0 & 0 & 2 & & & & \\
0 & -1 & 0 & 4 & & & \\
0 & 0 & -4 & 0 & 8 & & \\
0 & 1 & 0 & -12 & 0 & 16 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$$\sum_{n \geq 0} U_n(t)x^n = T\left(\frac{1}{2}\middle|\frac{1}{2} + \frac{1}{2}x^2\right)\left(\frac{1}{1 - xt}\right) = \frac{1}{1 + x^2 - 2xt}$$

The Chebyshev polynomials of the first kind:

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
In general, for $n \geq 2$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

We first produce a small perturbation and consider: $(\widetilde{T}(x))_{n\in\mathbb{N}}$ where $\widetilde{T}_0(x) = \dfrac{1}{2}$ and $\widetilde{T}_n(x) = T_n(x)$ for every $n \geq 1$.

$\widetilde{T}_0(x) = \frac{1}{2}$
$\widetilde{T}_1(x) = 2x\widetilde{T}_0(x)$
$\widetilde{T}_2(x) = 2x\widetilde{T}_1(x) - \widetilde{T}_0(x) - \frac{1}{2}$
and for $n \geq 3$

$$\widetilde{T}_n(x) = 2x\widetilde{T}_{n-1}(x) - \widetilde{T}_{n-2}(x)$$

$$\widetilde{T} = T\left(\frac{1}{4} - \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}x^2\right)$$

$$\left(\begin{array}{c|cccccc}
\frac{1}{4} & & & & & & \\
0 & \frac{1}{2} & & & & & \\
\frac{1}{4} & 0 & 1 & & & & \\
0 & -1 & 0 & 2 & & & \\
0 & 0 & -3 & 0 & 4 & & \\
0 & 1 & 0 & -8 & 0 & 8 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$$\sum_{n\geq 0} \widetilde{T}_n(t)x^n = T\left(\frac{1}{4} - \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}x^2\right)\left(\frac{1}{1-tx}\right) = \frac{1}{2}\frac{1-x^2}{1+x^2-2tx}$$

$$\sum_{n\geq 0} T_n(t)x^n = \frac{1}{2} + \sum_{n\geq 0} \widetilde{T}_n(t)x^n$$

$$\sum_{n\geq 0} T_n(t)x^n = \frac{1-tx}{1+x^2-2tx}$$

$$T\left(\frac{1}{4}-\frac{1}{4}x^2\middle|\frac{1}{2}+\frac{1}{2}x^2\right)=T\left(\frac{1}{2}-\frac{1}{2}x^2\middle|1\right)T\left(\frac{1}{2}\middle|\frac{1}{2}+\frac{1}{2}x^2\right)$$

So, symbolically

$$\begin{pmatrix} \widetilde{T}_0(x) \\ \widetilde{T}_1(x) \\ \widetilde{T}_2(x) \\ \widetilde{T}_3(x) \\ \widetilde{T}_4(x) \\ \widetilde{T}_5(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & & & & & & \\ 0 & \frac{1}{2} & & & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & & & \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & & \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} U_0(x) \\ U_1(x) \\ U_2(x) \\ U_3(x) \\ U_4(x) \\ U_5(x) \\ \vdots \end{pmatrix}$$

and consequently

$$\widetilde{T}_n(x) = -\frac{1}{2}U_{n-2}(x) + \frac{1}{2}U_n(x)$$

or

$$2\widetilde{T}_n(x) = U_n(x) - U_{n-2}(x)$$

and then

$$2T_n(x) = U_n(x) - U_{n-2}(x),\ \ n \geq 3$$

Proposition. Let $f = \sum_{n \geq 0} f_n x^n$, $g = \sum_{n \geq 0} g_n x^n$ be two power series such that $f_0 \neq 0$, $g_0 \neq 0$. Suppose that $(p_n(x))_{n \in \mathbb{N}}$ is the associated polynomial sequence of the Riordan array $T(f \mid g)$, then

(a) If $(q_n(x))_{n \in \mathbb{N}}$ is the associated sequence to $T(fg \mid g)$ we obtain

$$q_n(x) = x p_{n-1}(x) + f_n \quad \text{if} \quad n \geq 1$$

and $q_0(x) = f_0$.

(b) If $(r_n(x))_{n \in \mathbb{N}}$ is the associated polynomial sequence to $T\left(\frac{f}{g}\middle| g\right)$ then

$$r_{n-1}(x) = \frac{p_n(x) - p_n(0)}{x} \quad \text{for} \quad n \geq 1$$

Corollary Suppose $g = \sum_{n \geq 0} g_n x^n$ with $g_0 \neq 0$. Let $(p_n(x))_{n \in \mathbb{N}}$ be the pol. seq. ass. to $T(1 \mid g)$

$(q_n(x))_{n \in \mathbb{N}}$ that associated to $T(g \mid g)$.

$$q_n(x) = x p_{n-1}(x) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad q_0(x) = 1$$

Example: the Morgan-Voyce polynomials

$$T(1 \mid (1-x)^2) \qquad (B_n(x))$$

$$T(1-x|(1-x)^2) \qquad (b_n(x))$$

$$B_n(x) - B_{n-1}(x) = b_n(x)$$

$$b_n(x) - b_{n-1}(x) = xB_{n-1}(x)$$

$$T(1-x|1)T(1|(1-x)^2) = T(1-x|(1-x)^2)$$

$$T(1-x|1)T(1-x|(1-x)^2) = T((1-x)^2|(1-x)^2)$$

$(p_n(x))$ as the family of polynomials associated to $T(f \mid g)$,

$(q_n(x))$ the family of polynomials associated to each of the matrix products.

Moreover a, b are constant series with $b \neq 0$:

$$T(a \mid 1)T(f \mid g) = T(af \mid g), \text{ then } q_n(x) = ap_n(x)$$

$$T(1 \mid b)T(f \mid g) = T\left(f\left(\frac{x}{b}\right) \mid bg\left(\frac{x}{b}\right)\right), \text{ then}$$

$$q_n(x) = \frac{1}{b^{n+1}} p_n(x)$$

$$T(f \mid g)T(a \mid 1) = T(af \mid g), \text{ then } q_n(x) = ap_n(x)$$

$$T(f \mid g)T(1 \mid b) = T(f \mid bg), \text{ then } q_n(x) = \frac{1}{b} p_n\left(\frac{x}{b}\right)$$

Proposition. Let $T(f \mid g)$ and $T(l \mid m)$ be two element of the Riordan group. Suppose that $(p_n(x))$ and $(q_n(x))$ are the corresponding associated families of polynomials.

$$T(l \mid m) = T(\gamma \mid \alpha + \beta x)T(f \mid g)T(c \mid a + bx)$$

where $\alpha, \gamma, a, c \neq 0$. Then

$$q_n(x) = \frac{\gamma c}{\alpha a}\left(\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{\beta}{\alpha}\right)^{n-k}\frac{1}{\alpha^k}p_k\left(\frac{x-b}{a}\right)\right)$$

Example: Fibonacci and Pell polynomials

$T(1 \mid 1 - x^2)$ Fibonacci $\qquad T\left(\frac{1}{2}\middle|\frac{1}{2} - \frac{1}{2}x^2\right)$ Pell

$$T\left(\frac{1}{2}\middle|\frac{1}{2} - \frac{1}{2}x^2\right) = T\left(\frac{1}{2}\middle|1\right)T(1|1 - x^2)T\left(1\middle|\frac{1}{2}\right)$$

$$P_n(x) = F_n(2x)$$

Example: Fibonacci and Chebychev of second kind polynomials

$$T\left(\frac{1}{2}\middle|\frac{1}{2}(1 + x^2)\right) = T\left(\frac{1}{2}\middle| -i\right)T\left(1\middle|1 - x^2\right)T\left(1\middle|\frac{i}{2}\right)$$

$$U_n(x) = i^n F_n(-2ix)$$

Example: Fermat and Chebychev of second kind polynomials

The Fermat polynomials are $\mathcal{F}_0(x) = 1$, $\mathcal{F}_1(x) = 3x$ and for $n \geq 2$

$$\mathcal{F}_n(x) = 3x\mathcal{F}_{n-1}(x) - 2\mathcal{F}_{n-2}(x)$$

$$T\left(\frac{1}{3}\Big|\frac{1}{3} + \frac{2}{3}x^2\right)$$

$$\left(\begin{array}{c|cccccccc}
\frac{1}{3} & & & & & & & & \\
0 & 1 & & & & & & & \\
0 & 0 & 3 & & & & & & \\
0 & -2 & 0 & 9 & & & & & \\
0 & 0 & -12 & 0 & 27 & & & & \\
0 & 4 & 0 & -54 & 0 & 81 & & & \\
0 & 0 & 36 & 0 & -216 & 0 & 243 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$\mathcal{F}_0(x) = 1$
$\mathcal{F}_1(x) = 3x$
$\mathcal{F}_2(x) = -2 + 9x^2$
$\mathcal{F}_3(x) = -12x + 27x^3$
$\mathcal{F}_4(x) = 4 - 54x^2 + 81x^4$
$\mathcal{F}_5(x) = 36x - 216x^3 + 243x^5$

$$T\left(\frac{1}{3}\Big|\frac{1}{3}(1+2x^2)\right) = T\left(1\Big|\frac{1}{\sqrt{2}}\right) T\left(\frac{1}{2}\Big|\frac{1}{2}(1+x^2)\right) T\left(\frac{2}{3}\Big|\frac{2\sqrt{2}}{3}\right)$$

$$\mathcal{F}_n(x) = (\sqrt{2})^n U_n\left(\frac{3x}{2\sqrt{2}}\right)$$

Generalized Appell polynomials (Boas-Buck, 1964)

Proposition: $(s_n(x))$ is a *family of generalized Appell polynomials* iff $\exists\ f, g, h \in \mathbb{K}[[x]]$, $f = \sum_{n\geq 0} f_n x^n$, $g = \sum_{n\geq 0} g_n x^n$ and $h(x) = \sum_{n\geq 0} h_n x^n$ with $f_0, g_0 \neq 0$, and $h_n \neq 0$ for all n such that

$$T(f \mid g)h(tx) = \sum_{n\geq 0} s_n(t)x^n$$

Moreover,

$$s_n(x) = (p_n \star h)(x)$$

where $(p_n(x))$ is a.p.s $T(f \mid g)$

So

$$s_n(x) = \sum_{k=0}^{n} p_{n,k} h_k x^k$$

$$\sum_{n\geq 0} s_n(t)x^n = \sum_{n\geq 0} (p_n \star h)(t)x^n = \frac{f(x)}{g(x)} h\left(t\frac{x}{g(x)}\right)$$

Example: The Sheffer polynomials.

$$T(f \mid g)(e^{tx}) = \sum_{n \geq 0} S_n(t) x^n$$

$$\sum_{n \geq 0} S_n(t) x^n = A(x) e^{tH(x)}$$

where $A = \sum_{n \geq 0} A_n x^n, H = \sum_{n \geq 1} H_n x^n$ with $A_0 \neq 0$, $H_1 \neq 0$.

$$T\left(\frac{xA}{H}\middle|\frac{x}{H}\right)$$

$$S_n(x) = p_n(x) \star e^x$$

$$S_n(x) = \sum_{k=0}^{n} \frac{p_{n,k}}{k!} x^k$$

if $p_n(x) = \sum_{k=0}^{n} p_{n,k} x^k$.

WARNING: Often called a Sheffer sequence to the sequence $(n! S_n(x))_{n \in \mathbb{N}}$ where $(S_n(x))_{n \in \mathbb{N}}$ is our Sheffer sequence.

Example: The Brenke polynomials Following Boas-Buck, $(B_n(x))$ is in the class of Brenke polynomials if

$$T(f \mid 1)(h(tx)) = \sum_{n \geq 0} B_n(t)x^n$$

Some particular cases are:

$$T(f \mid 1)(e^{tx}) = \sum_{n \geq 0} A_n(t)x^n$$

where $(A_n(x))$ are the Appell polynomials of f.

$$T(f \mid 1)\left(\frac{1}{1-tx}\right) = \sum_{n \geq 0} T_n^*(t)x^n$$

where (T_n^*) are the reversed Taylor polynomial of f.

Example: Pidduck and Mittag-Leffler polynomials. Consider $(\mathcal{P}_n(x))$

$$\sum_{n\geq 0} \mathcal{P}_n(t)x^n = T\left(\frac{x}{(1-x)\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)(e^{tx})$$

$$\widetilde{P}_n(x) = n!\mathcal{P}_n(x)$$

If $(M_n(x))$ is given by the formula:

$$\sum_{n\geq 0} M_n(t)x^n = T\left(\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)(e^{tx})$$

$$\widetilde{M}_n(x) = n!M_n(x)$$

Note that:

$$T\left(\frac{x}{(1-x)\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right) = T\left(\frac{1}{1-x}\middle|1\right) T\left(\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\middle|\frac{x}{\log\left(\frac{1+x}{1-x}\right)}\right)$$

$$\mathcal{P}_n(x) = \sum_{k=0}^{n} M_k(x)$$

or equivalently

$$\widetilde{P}_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n-k)!\widetilde{M}_k(x)$$

Example: The Laguerre polynomials.

$$T(-1 \mid x-1)(e^{tx}) = T(1 \mid 1-x)T(-1 \mid -1)(e^{tx}) =$$

$$= T(1 \mid 1-x)(e^{-tx}) = \sum_{k=0}^{n} L_n(t)x^n$$

General term:

$$L_n(x) = p_n(x) \star e^{-x} = \sum_{k=0}^{n} \binom{n}{k} x^k \star \sum_{k \geq 0} \frac{(-1)^k}{k!} x^k = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \binom{n}{k} x^k$$

Recurrence relation:

$$L_n(x) = xL_{n-1}(x) \star (-\log(1-x)) + L_{n-1}(x)$$

If $k \geq 1$,

$$L_{n,k} = L_{n-1,k} - \frac{1}{k} L_{n-1,k-1}$$

$$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x)$$

$$L_n'(x) = -\sum_{k=0}^{n-1} L_k(x)$$

Example: The Hermite polynomials.

$$\sum_{n\geq 0} H_n(t)x^n = T\left(\frac{1}{2e^{x^2}}\middle|\frac{1}{2}\right)(e^{tx}) =$$

$$= T\left(\frac{1}{e^{x^2}}\middle|1\right)T\left(\frac{1}{2}\middle|\frac{1}{2}\right)(e^{tx}) = T\left(\frac{1}{e^{x^2}}\middle|1\right)(e^{2tx}) = e^{2tx-x^2}$$

$$H_n(x) = xH_{n-1}(x) \star \widehat{h}(x) + f_n$$

$$\widehat{h}(x) = \sum_{n\geq 1}\frac{2}{n}x^n = -2\log(1-x)$$

$$f_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!}, & \text{if } n \text{ is even.} \end{cases}$$

$$H_{n,k} = \frac{2}{k}H_{n-1,k-1}$$

$H'_n(x) = 2H_{n-1}(x)$ or $\widetilde{H}'_n(x) = 2n\widetilde{H}_{n-1}(x)$.

$$H_{2m}(x) = \sum_{j=0}^{m}\frac{(-1)^{m-j}2^{2j}}{(m-j)!(2j)!}x^{2j}$$

$$H_{2m+1}(x) = \sum_{j=0}^{m}\frac{(-1)^{m-j}2^{2j+1}}{(m-j)!(2j+1)!}x^{2j+1}$$

$$\widetilde{H}_n(-x) = (-1)^n\widetilde{H}_n(x)$$

Conclusions:

We can use the structure of Riordan group to obtain information about families of sequences of polynomials. We can study each subgroup of Riordan group and its families of polynomials associated, for example:

$T(f \mid 1)h(xt)$ and the Brenke polynomials.

$T(g \mid g)(e^{xt})$ and convolution polynomials or polynomials of binomial type.

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